۱. جواب عمومی معادله دیفرانسیل زیر را در همسایگی نقطه $x=0$ به ازای ریشه بزرگتر معادله شاخص به دست آورید.

$$x^2(1+x)y''+x(x-4)y'+4y=0$$

$$y''+\frac{x-4}{x\,(1+x)}y'+\frac{4}{x^2(1+x)}y=0$$

$$p(x)=\frac{x-4}{x\,(1+x)} \qquad \Rightarrow \qquad x\,p(x)=\frac{x-4}{1+x} \qquad \Rightarrow \qquad p_0=-4$$

$$q(x)=\frac{4}{x^2(1+x)} \qquad \Rightarrow \qquad x^2\,q(x)=\frac{4}{1+x} \qquad \Rightarrow \qquad q_0=4$$

$$r(r-1)+p_0\,r+q_0=0 \quad \Rightarrow \quad r(r-1)-4\,r+4=0 \quad \Rightarrow \quad r^2-5\,r+4=0 \quad \Rightarrow \quad r_1=4, \quad r_2=1$$

$$y=x^4\sum_{n=0}^{\infty}a_nx^n=\sum_{n=0}^{\infty}a_nx^{n+4} \qquad y'=\sum_{n=0}^{\infty}(n+4)\,a_n\,x^{n+3} \qquad y''=\sum_{n=0}^{\infty}(n+3)(n+4)\,a_n\,x^{n+2}$$

$$x^2(1+x)\sum_{n=0}^{\infty}(n+3)(n+4)\,a_n\,x^{n+2}+x(x-4)\sum_{n=0}^{\infty}(n+4)\,a_n\,x^{n+3}+4\sum_{n=0}^{\infty}a_nx^{n+4}=0$$

$$\sum_{n=0}^{\infty}(n+3)(n+4)\,a_n\,x^{n+4}+\sum_{n=0}^{\infty}(n+3)(n+4)\,a_n\,x^{n+5}+\sum_{n=0}^{\infty}(n+4)\,a_n\,x^{n+5}-4\sum_{n=0}^{\infty}(n+4)\,a_n\,x^{n+4}+4\sum_{n=0}^{\infty}a_nx^{n+4}=0$$

$$\sum_{n=0}^{\infty}[(n+3)(n+4)-4(n+4)+4]\,a_n\,x^{n+4}+\sum_{n=0}^{\infty}[(n+3)(n+4)+(n+4)]\,a_n\,x^{n+5}=0$$

$$\sum_{n=0}^{\infty}[(n-1)(n+4)+4]\,a_n\,x^{n+4}+\sum_{n=0}^{\infty}(n+4)^2\,a_n\,x^{n+5}=0$$

$$n+5=m+4 \qquad \Rightarrow \qquad n=m-1$$

$$\sum_{n=0}^{\infty}[(n-1)(n+4)+4]\,a_n\,x^{n+4}+\sum_{n=1}^{\infty}(n+3)^2\,a_{n-1}\,x^{n+4}=0$$

$$\sum_{n=1}^{\infty}\{[(n-1)(n+4)+4]a_n+(n+3)^2\,a_{n-1}\}\,x^{n+4}=0$$

$$[(n-1)(n+4)+4]a_n+(n+3)^2\,a_{n-1}=0 \qquad \Rightarrow \qquad a_n=-\frac{(n+3)^2}{n^2+3n}\;a_{n-1}=-\frac{(n+3)^2}{n(n+3)}\;a_{n-1}=-\frac{n+3}{n}\;a_{n-1}$$

$$\Rightarrow \qquad a_n=-(1+\frac{3}{n})\;a_{n-1} \qquad n=1,2,3,\dots$$

$$a_1=-4\;a_0$$

$$a_2=-\frac{5}{2}\;a_1=-\frac{5}{2}\,(-4)\,a_0=10\,a_0$$

$$a_3=-2\;a_2=-2\,(10)\,a_0=-20\,a_0$$

$$a_4=-\frac{7}{4}\;a_3=-\frac{7}{4}\,(-20)\,a_0=35\,a_0$$

$$y=x^4\sum_{n=0}^{\infty}a_nx^n \qquad \Rightarrow \qquad y=a_0\,x^4(1-4\,x+10\,x^2-20\,x^3+35\,a_0\,x^4+\cdots)$$

۲. با استفاده از تغییر متغیر $y=\frac{u}{x}$ معادله دیفرانسیل زیر را حل کنید.

$$x\,y''+3\,y'+xy=0,$$

$$y=\frac{u}{x} \quad\Rightarrow\quad y'=\frac{u'x-u}{x^2} \quad\Rightarrow\quad y''=\frac{(u''x+u'-u')\,x^2-2x(u'x-u)}{x^4}=\frac{u''\,x^3-2x(u'x-u)}{x^4}=\frac{u''\,x^2-2(u'x-u)}{x^3}$$

$$x\,y''+3\,y'+xy=0 \quad\Rightarrow\quad \frac{u''\,x^2-2(u'x-u)}{x^2}+3\,\frac{u'x-u}{x^2}+u=0$$

$$\Rightarrow\quad x^2\,u''-2\,x\,u'+2u+3\,u'x-3u+x^2\,u=0$$

$$\Rightarrow\quad x^2\,u''+x\,u'-u+x^2\,u=0$$

$$\Rightarrow\quad x^2\,u''+x\,u'+(x^2-1)\,u=0$$

$$p=1\in\mathbb{Z} \quad\Rightarrow\quad u(x)=c_1\,J_1(x)+\,c_2\,Y_1(x)$$

$$\Rightarrow\quad y(x)=\frac{c_1}{x}\,J_1(x)+\frac{c_2}{x}\,Y_1(x)$$

۳. مطلوب است محاسبه موارد زیر:

الف)

$$L^{-1}\{e^{-2s}\ ln\ (\frac{s^2+1}{s^2})\}$$

$$L^{-1}\{e^{-2s}\ ln\ (\frac{s^2+1}{s^2})\} = H(t-2)\, f(t-2)$$

$$f(t) = L^{-1}\left\{ln\left(\frac{s^2+1}{s^2}\right)\right\} \quad \Rightarrow \quad L\{f(t)\} = ln\left(\frac{s^2+1}{s^2}\right) \quad \Rightarrow \quad L\{tf(t)\} = \frac{-d}{ds}\ ln\ (\frac{s^2+1}{s^2})$$

$$\Rightarrow \quad L\{tf(t)\} = \frac{-d}{ds}[ln(s^2+1) - 2\ln(s)] = -\frac{2s}{s^2+1} + 2\frac{1}{s}$$

$$\Rightarrow \quad tf(t) = 2L^{-1}\left\{\frac{1}{s}\right\} - 2L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$\Rightarrow \quad tf(t) = 2 - 2\cos(t)$$

$$\Rightarrow \quad f(t) = \frac{2(1-\cos(t))}{t}$$

$$\Rightarrow \quad L^{-1}\{e^{-2s}\ ln\ (\frac{s^2+1}{s^2})\} = H(t-2)\ \frac{2(1-\cos(t-2))}{t-2}$$

(ب)

$$\int_0^\infty e^{-t}\ \frac{(1-\cos t)}{t}\ dt$$

$$f(t) = e^{-t}\ (1-\cos t)$$

$$F(s) = L\{e^{-t}\ (1-\cos t)\} = \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}$$

$$\int_{t=0}^{\infty} \frac{f(t)}{t}\ dt = \int_{s=0}^{\infty} F(s)\ ds \qquad \Rightarrow \qquad \int_{t=0}^{\infty} e^{-t}\ \frac{(1-\cos t)}{t}\ dt = \int_{s=0}^{\infty} \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}\ ds$$

$$= \int_{u=1}^{\infty} \frac{1}{u} - \frac{u}{u^2+1}\ du \qquad (u = s+1)$$

$$= [ln(\frac{u}{\sqrt{u^2+1}})]_{u=1}^{\infty}$$

$$= ln(1) - ln(\frac{1}{\sqrt{2}})$$

$$= \frac{ln(2)}{2}$$

۴. جواب مساله مقدار اولیه زیر را بیابید:

$$2\,y' = -y + e^{x} - \int_0^x e^{(x-t)}\ y'(t)\ dt, \qquad\qquad y(0) = 1$$

$$\Rightarrow\quad 2L\{y'\} = -L\{y\} + L\{e^{x}\} - L\left\{\int_0^x e^{(x-t)}\ y'(t)\ dt\right\}$$

$$\Rightarrow\quad 2\,(s\,Y(s) - 1) = -Y(s) + \frac{1}{s-1} - L\{e^{x}\}\,L\{y'(x)\}$$

$$\Rightarrow\quad 2\,(s\,Y(s) - 1) = -Y(s) + \frac{1}{s-1} - \frac{1}{s-1}(\,s\,Y(s) - 1)$$

$$\Rightarrow\quad \left(2\,s + 1 + \frac{s}{s-1}\right)Y(s) = \frac{1}{s-1} + \frac{1}{s-1} + 2$$

$$\Rightarrow\quad (2s^2 + s - 2s - 1 + s)Y(s) = 1 + 1 + 2s - 2$$

$$\Rightarrow\quad (2s^2 - 1)\,Y(s) = 2s$$

$$\Rightarrow\quad Y(s) = \frac{2s}{2s^2 - 1} \qquad \Rightarrow \qquad Y(s) = \frac{s}{s^2 - \frac{1}{2}}$$

$$\Rightarrow\quad y(x) = \cosh\left(\frac{x}{\sqrt{2}}\right)$$